The framework of the enclosure method with dynamical data and its applications

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Abstract

The aim of this paper is to establish the *framework* of the enclosure method for some class of inverse problems whose governing equations are given by parabolic equations with discontinuous coefficients.

The framework is given by considering a concrete inverse initial boundary value problem for a parabolic equation with discontinuous coefficients. The problem is to extract information about the location and shape of unknown inclusions embedded in a known isotropic heat conductive body from a set of the input heat flux across the boundary of the body and output temperature on the same boundary. In the framework the original inverse problem is reduced to an inverse problem whose governing equation have a *large parameter*. A list of requirements which enables one to apply the enclosure method to the reduced inverse problem is given.

Two new results which can be considered as the applications of the framework are given. In the first result the background conductive body is assumed to be *homogeneous* and a family of explicit *complex* exponential solutions are employed. Second an application of the framework to inclusions in an isotropic *inhomogeneous* heat conductive body is given. The main problem is the construction of the special solution of the governing equation with a large parameter for the background inhomogeneous body required by the framework. It is shown that, introducing another parameter which is called the *virtual slowness* and making it *sufficiently large*, one can construct the required solution which yields an extraction formula of the *convex hull* of unknown inclusions in a known isotropic inhomogeneous conductive body.
AMS: 35R30
KEY WORDS: inverse initial boundary value problem, enclosure method, heat equation, thermal imaging, modified Helmholtz equation, convex hull

1 Introduction

The aim of this paper is to establish the *framework* of the enclosure method [9] for possible application to some class of inverse problems whose governing equations are given by parabolic equations with discontinuous coefficients.

The framework is given by considering a concrete inverse initial boundary value problm for a parabolic equation with discontinuous coefficients. The problem is to extract information about the location and shape of unknown inclusions embedded in a known isotropic

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heat conductive body from a set of the input heat flux across the boundary of the body and output temperature on the same boundary.

Let Ω be a bounded domain of $\mathbf{R}^n$, $n = 2, 3$ with a smooth boundary. We denote the unit outward normal vectors to $\partial\Omega$ by the symbol ν. Let T be an arbitrary *fixed* positive number.

Given $f = f(x,t)$, $(x,t) \in \partial\Omega\times\,]0,\,T[$ let $u = u_f(x,t)$ be the solution of the initial boundary value problem for the parabolic equation:

$$
\begin{aligned}
&\partial_t u - \nabla\cdot\gamma\nabla u = 0 \text{ in}\,\Omega\times\,]0,\,T[,\\
&\gamma\nabla u\cdot\nu = f \text{ on}\,\partial\Omega\times\,]0,\,T[,\\
&u(x,0) = 0 \text{ in}\,\Omega,
\end{aligned}
\tag{1.1}
$$

where $\gamma = \gamma(x) = (\gamma_{ij}(x))$ satisfies

(G1) for each $i, j = 1, \cdots, n$ $\gamma_{ij}(x)$ is real and satisfies $\gamma_{ij}(x) = \gamma_{ij}(x) \in L^\infty(\Omega)$;

(G2) there exists a positive constant C such that $\gamma(x)\xi\cdot\xi \ge C|\xi|^2$ for all $\xi \in \mathbf{R}^n$ and a. e. $x \in \Omega$.

See [3] for the notion of the weak solution. This paper is concerned with the extraction of information about "discontinuity" of γ from u and $\gamma\nabla u\cdot\nu$ on $\partial\Omega\times]0,T[$ for some f and an arbitrary fixed $T < \infty$. However, we do not consider completely general γ. Instead we assume that there exists an open set D with a smooth boundary such that $\overline{D} \subset \Omega$ and $\gamma(x)$ a.e. $x \in \Omega\setminus D$ coincides with a smooth positive function $\gamma_0(x)$ of $x \in \overline{\Omega}$ and satisfies one of the following two conditions:

(A1) there exists a positive constant C' such that $-(\gamma(x) - \gamma_0(x)I_n)\xi\cdot\xi \ge C'|\xi|^2$ for all $\xi \in \mathbf{R}^n$ and a.e. $x \in D$;

(A2) there exists a positive constant C' such that $(\gamma(x) - \gamma_0(x)I_n)\xi\cdot\xi \ge C'|\xi|^2$ for all $\xi \in \mathbf{R}^n$ and a.e. $x \in D$.

Write $h(x) = \gamma(x) - \gamma_0(x)I_n$ a.e. $x \in D$.
We consider

Inverse Problem 1.1. Fix a $T > 0$. Assume that both D and h are *unknown* and that γ_0 is *known*. Extract information about the location and shape of D from a set of the pair of temperature $u_f(x,t)$ and heat flux $f(x,t)$ for $(x,t) \in \partial\Omega\times\,]0,\,T[$.

The D is a model of the union of unknown *inclusions* where the heat conductivity is anisotropic, different from that of the surrounding *inhomogeneous* isotropic conductive medium. The problem is a mathematical formulation of a typical inverse problem in thermal imaging. Note that in [4] a uniqueness theorem with *infinitely many* f for Inverse Problem 1.1 has been established provided h has the form bI_n with a smooth function b on $\overline{D}$. Thus the point is to give a concrete procedure or formula which yield information

about the location and shape of D. Note that when $n=1$, there are some results: the procedure in [1, 2] with *infinitely many* f and the formula of Theorem 2.1 in [10] with a *single* f.

In [14] we considered Inverse Problem 1.1 in the case when $\gamma_0\equiv 1$ and $n=3$ and gave *four* extraction formulae of some information including the convex hull of D. In Subsection 1.3 we will reconsider three results of those which employ infinitely many f, from the view point of the framework given here.

Note that in this paper we do not consider the *single input* case. Formulae with a single f see Theorem 1.1 in [14] and [15] for a cavity which is the extremal case $\gamma\equiv 0$ in D. Those can be considered as some kind of extension of the enclosure method for elliptic equations with a single Cauchy data started in [8] to the parabolic equations.

1.1 A reduction to an inverse boundary value problem with a parameter-A general framework

Define

$$w_f(x,\tau)=\int_0^T e^{-\tau t}u_f(x,t)dt,\ x\in\Omega,\ \tau>0.$$

The $w=w_f$ satisfies

$$\begin{aligned}(\nabla\cdot\gamma\nabla-\tau)w&=e^{-\tau T}u_f(x,T)\ \text{in}\,\Omega,\\ \gamma\nabla w\cdot\nu&=\int_0^T e^{-\tau t}f(x,t)dt\ \text{for}\,x\in\partial\Omega.\end{aligned}\tag{1.2}$$

This motivates a formulation of the reduced problem given below.

Given $F(x,\tau)$ and $g(x,\tau)$ with let $w=w(x,\tau)$ be the solution of

$$\begin{aligned}(\nabla\cdot\gamma\nabla-\tau)w&=e^{-\tau T}F(x,\tau)\ \text{in}\,\Omega,\\ \gamma\nabla w\cdot\nu&=g(x,\tau)\ \text{on}\,\partial\Omega.\end{aligned}\tag{1.3}$$

Inverse problem 1.2. Assume that $F(x,\tau)$, D and h are all unknown and that γ_0 is known. Extract information about the location and shape of D from a set of the pair of $w(x,\tau)$ and $g(x,\tau)$ for $x\in\partial\Omega$, $\tau>0$.

For this problem we propose the following general framework which reduces the problem to construct a family of special solutions of an equation coming from the background body.

Theorem 1.1. *Assume that: there exist constants C_1 and μ_1 such that, as $\tau\longrightarrow\infty$*

$$\|F(\,\cdot\,,\tau)\|_{L^2(\Omega)}=O(e^{C_1\tau}\tau^{\mu_1});\tag{1.4}$$

we have a family (v_τ) indexed with $\tau\ge\tau_0>0$ of solutions of the equation

$$\nabla\cdot\gamma_0\nabla v-\tau v=0\ \textit{in}\ \Omega\tag{1.5}$$

satisfying the conditions, for some constants μ_2, μ_3, μ_4, C_2 and C_3

$$\|\nabla v_\tau\|_{L^2(D)}=O(e^{C_2\tau}\tau^{\mu_2}),\tag{1.6}$$

$$\|\nabla v_\tau\|_{L^2(D)} \geq C'' e^{C_2\tau}\tau^{\mu_3}, \tag{1.7}$$

$$\|v_\tau\|_{H^1(\Omega)} = O(e^{C_3\tau}\tau^{\mu_4}). \tag{1.8}$$

Let $g = g(x,\tau)$ be a function of $x \in \partial\Omega$ having the form

$$g = \Psi(\tau)\,\gamma_0 \frac{\partial v_\tau}{\partial \nu}|_{\partial\Omega},$$

where Ψ satisfies the conditions, for constants μ and μ'

$$\liminf_{\tau\longrightarrow\infty} \tau^\mu |\Psi(\tau)| > 0 \tag{1.9}$$

and

$$|\Psi(\tau)| = O(\tau^{\mu'}). \tag{1.10}$$

Let w be the solution of (1.3). If T satisfies

$$T > C_1 + C_3 - 2C_2, \tag{1.11}$$

then

$$\lim_{\tau\longrightarrow\infty} \frac{1}{2\tau} \log\left|\int_{\partial\Omega}\left(g\overline{v}_\tau - w\gamma_0\frac{\partial\overline{v}_\tau}{\partial\nu}\right)dS\right| = C_2. \tag{1.12}$$

Remark 1.1. It follows from (1.6) and (1.7) that

$$\lim_{\tau\longrightarrow\infty} \frac{1}{\tau}\log\|\nabla v_\tau\|_{L^2(D)} = C_2.$$

This is the meaning of C_2 which is uniquely determined by $\|\nabla v_\tau\|_{L^2(D)}$ with all $\tau >> \tau_0$.
Remark 1.2. It follows from (1.7) and (1.8) that $C_3 \geq C_2$.
Remark 1.3. From the proof one obtains also the order of the convergence of the formula (1.12):

$$\frac{1}{2\tau}\log\left|\int_{\partial\Omega}\left(g\overline{v}_\tau - w\gamma_0\frac{\partial\overline{v}_\tau}{\partial\nu}\right)dS\right| = C_2 + O\left(\frac{\log\tau}{\tau}\right).$$

This is important for a suitable choice of τ in the case when the data is noisy.

Here we present an application of Theorem 1.1 to the case when $\gamma_0 \equiv 1$. Let $c > 0$ and $\tau \geq \tau_0 = c^{-2}$. Let $\omega, \omega^\perp \in S^{n-1}$, $n \geq 2$ and satisfy $\omega\cdot\omega^\perp = 0$. Set

$$z = c\tau\left(\omega + i\sqrt{1-\frac{1}{c^2\tau}}\,\omega^\perp\right). \tag{1.13}$$

z satisfies

$$z\cdot z = \tau. \tag{1.14}$$

We observe that

$$(\triangle - \tau)e^{x\cdot z} = 0. \tag{1.15}$$

Thus we take the family $(v_\tau)_{\tau\geq\tau_0}$ in Theorem 1.1

$$v_\tau(x) = v(x;z) = e^{x\cdot z}.$$

Note that:

• it follows from (1.15) that the function $e^{-\tau t}v(x;z)$ of (x,t) satisfies the *backward heat equation*

$$(\triangle+\partial_t)(e^{-\tau t}v(x;z))=0.$$

• the absolute value of $e^{-\tau t}v(x;z)$ coincides with $e^{-\tau(t-cx\cdot\omega)}$ and this is a solution of the *wave equation* with the propagation speed $1/c$. By this reason we call this c the *virtual slowness*. See also [10] for this interpretation.

Recall the support functions of D and Ω:

$$h_D(\omega)=\sup_{x\in D}x\cdot\omega,\ \ h_\Omega(\omega)=\sup_{x\in\Omega}x\cdot\omega.$$

We have $h_D(\omega)<h_\Omega(\omega)$ for all ω since $\overline{D}\subset\Omega$.

Applying Theorem 1.1 to (1.2), we obtain the following corollary.

Corollary 1.1. *Assume that $\gamma_0\equiv 1$. Let f be the function of $(x,t)\in\partial\Omega\times]0,\,T[$ having a parameter $\tau>0$ defined by the equation*

$$f(x,t)=\frac{\partial v_\tau}{\partial\nu}(x)\varphi(t),\tag{1.16}$$

where a real-valued function $\varphi\in L^2(0,\,T)$ satisfying the condition: there exists $\mu\in\mathbf{R}$ such that

$$\liminf_{\tau\longrightarrow\infty}\tau^\mu\left\vert\int_0^T e^{-\tau t}\varphi(t)dt\right\vert>0.\tag{1.17}$$

Let $u_f=u_f(x,t)$ be the weak solution of (1.1) for $f=f(x,t)$. If T satisfies

$$T>2c(h_\Omega(\omega)-h_D(\omega)),\tag{1.18}$$

then

$$\lim_{\tau\longrightarrow\infty}\frac{1}{2\tau}\log\left\vert\int_{\partial\Omega}\int_0^T e^{-\tau t}\left(-\overline{v_\tau(x)}f(x,t;\tau)+u_f(x,t)\frac{\partial\overline{v_\tau}}{\partial\nu}(x)\right)dtdS\right\vert=h_D(\omega).\tag{1.19}$$

Remark 1.4. There is no restriction on the position of center of coordinates relative to Ω and D which affects on the sign of $h_D(\omega)$.

Since $-h_\Omega(-\omega)=\inf_{x\in\Omega}x\cdot\omega<\inf_{x\in D}x\cdot\omega\le h_D(\Omega)$, we have

$$h_\Omega(\omega)-h_D(\omega)<h_\Omega(\omega)+h_\Omega(-\omega).$$

Thus T in Corollary 1.1 can be *arbitrary small* by choosing a small known c in such a way that, for example,

$$c\le\frac{T}{2(h_\Omega(\omega)+h_\Omega(-\omega))}$$

since from this one gets (1.18). Therefore if one wants to estimate D from the direction ω by the formula (1.19) and the 'size' of Ω at the ω direction, that is the quantity $h_\Omega(\omega)+h_\Omega(-\omega)$, is too large compared with T (this is the most difficult case), the virtual slowness c in (1.13) should be chosen very small. This is one of the two roles of virtual slowness. In the next subsection we give another role of virtual slowness.

Let us explain how to deduce (1.19) from Theorem 1.1. Comparing (1.2) with (1.3), one knows that the g and F in Theorem 1.1 have the form

$$g(x,\tau)=\int_0^T e^{-\tau t}\varphi(t)dt\,\frac{\partial v_\tau}{\partial\nu}(x),\ x\in\partial\Omega,\ \tau>0 \tag{1.20}$$

and

$$F(x,\tau)=u_f(x,T). \tag{1.21}$$

Thus (1.9) is satisfied with the same μ as (1.17); (1.18) is satisfied with $\mu'=-1$. We have to know also an estimation of C_1 in (1.4) from above. From [3] it follows that

$$\|u(\,\cdot\,,T)\|_{L^2(\Omega)}=O(\|f\|_{L^2(0,T;H^{-1/2}(\partial\Omega))}) \tag{1.22}$$

and thus one can choose

$$C_1=ch_\Omega(\omega).$$

We have also

$$C_2=ch_D(\omega)<ch_\Omega(\omega)=C_3.$$

Since $C_1+C_3-2C_2=2c(h_\Omega(\omega)-h_D(\omega))$, (1.11) becomes (1.18).

Remark 1.5. Note that the choice (1.13) of z satisfying (1.14) goes back to [11] in which an application of the enclosure method for inverse source problems for the heat equations are given. The point of the choice is: the growing orders of $|z|$ and $z\cdot z$ as $\tau\longrightarrow\infty$ are same. See also [12] for an application to the so-called inverse heat conduction problem. In one-space dimensional case, in [10] instead of (1.13) the z having the form

$$z=c\tau\left(1+i\sqrt{1-\frac{1}{c^2\tau}}\right)$$

has been used. In this case $\text{Re}\,(z\cdot z)=\tau$. Formula (1.19) can be considered as an extension of a result in one-space dimensional case [10]. It means that if one could always *control* the initial temperature in the process of all possible measurements at the boundary to be zero, then one can extract the *convex hull* of unknown inclusions.

Since f is *complex-valued*, u_f on $\partial\Omega\times\,]0,\,T[$ can not be directly measured and should be computed from *real* data via the formula

$$u_f=u_{\text{Re}\,f}+iu_{\text{Im}\,f}. \tag{1.23}$$

This is a consequence of the zero initial data. Thus the zero initial data is essential for this procedure. Note also that since both $\text{Re}\,f$ and $\text{Im}\,f$ are highly oscillatory as $\tau\longrightarrow\infty$ with respect to the space variables, it will be difficult to prescribe those fluxes on the boundary directly. Instead one has to make use of the principle of superposition to *compute* the right-hand side of (1.23) on $\partial\Omega$ from *experimental data* which are generated by finite numbers of independent simpler input fluxes on $\partial\Omega$. Needles to say, for this procedure the zero initial data are also essential.

1.2 The case when γ_0 is not necessary constant.

It is possible to to extend Corollary 1.1 to the case when γ_0 is *not* necessary a constant. For simplicity of description assume that $\gamma_0 - 1 \in C_0^\infty(\mathbf{R}^n)$. We construct a special solution of the equation of (1.5) which has the form

$$v(x) \sim \frac{e^{x\cdot z}}{\sqrt{\gamma_0}}$$

as $\tau \longrightarrow \infty$, where z is given by (1.13).

Following [19], we make use of the change of the dependent variable formula (the Liouville transform):

$$\frac{1}{\sqrt{\gamma_0}}\nabla\cdot\gamma_0\nabla\left(\frac{1}{\sqrt{\gamma_0}}\cdot\right) = \triangle - V, \tag{1.24}$$

where

$$V = \frac{\triangle\sqrt{\gamma_0}}{\sqrt{\gamma_0}}. \tag{1.25}$$

We find the special solution of (1.5) having the form

$$v = \frac{e^{x\cdot z}}{\sqrt{\gamma_0}}(1+\epsilon_z),$$

where ϵ_z is a new unknown function. It follows from (1.24) and (1.14) that the equation for $\epsilon = \epsilon_z$ becomes

$$\left\{\triangle + 2z\cdot\nabla - \tau\left(\frac{1}{\gamma_0}-1\right) - V\right\}\epsilon = \tau\left(\frac{1}{\gamma_0}-1\right) + V. \tag{1.26}$$

Thus the problem is to construct a solution of equation (1.26) such that $\epsilon_z \approx 0$ in Ω as $\tau \longrightarrow \infty$. In general this is not an easy task because of the growing factor τ on the zeroth-order term in (1.26). However, we found that: if the virtual slowness c in (1.13) is *sufficiently large and fixed*, then one can construct such a solution for all large $\tau >> 1$ with an arbitrary small ϵ_z by using a combination of the Fourier transform and perturbation methods in [19] for the construction of the so-called complex geometrical optics solutions of the equation $\nabla\cdot\gamma_0\nabla v = 0$. Its precise description is the following second result.

Theorem 1.2. *Let $-1 < \delta < 0$ and $a, b \in C_0^\infty(\mathbf{R}^n)$. Given $\eta > 0$ there exist positive constants $C_j = C_j(a, b, \Omega, \delta, \eta)$, $j = 1, 2$ such that: if $c \geq C_1$ and $\tau \geq C_2$, then $c^2\tau > 1$ and there exists a unique $\epsilon_z \in L^2_\delta(\mathbf{R}^n)$ with z given by (1.13) such that*

$$(\triangle + 2z\cdot\nabla - \tau a - b)\epsilon_z = \tau a + b \text{ in}\, \mathbf{R}^n. \tag{1.27}$$

Moreover, $\epsilon_z|_\Omega$ can be identified with a function in $C^1(\overline{\Omega})$ and

$$\|\epsilon_z\|_{L^\infty(\Omega)} + \|\nabla\epsilon_z\|_{L^\infty(\Omega)} \leq \eta. \tag{1.28}$$

This theorem indicates the important role of the virtual slowness c when γ_0 is not necessary constant. It is not an accessary! To the best knowledge of the author this idea, that is, choosing a *large* c and fix, never been pointed out.

Having this theorem, we obtain a result which corresponds to Corollary 1.1.

Let $0 < \eta << 1$ and fix a $c \ge C_1$ in Theorem 1.2. Let $a = (1/\gamma_0 - 1)$ and b is given by (1.25). Let ϵ_z be the solution of (1.27) constructed in Theorem 1.2. Define

$$v_\tau(x) = \frac{e^{x\cdot z}}{\sqrt{\gamma_0(x)}}(1+\epsilon_z(x)),\ x \in \Omega,\ \tau \ge C_2. \tag{1.29}$$

The function $e^{-\tau t}v_\tau(x)$ satisfies the backward parabolic equation

$$(\nabla\cdot\gamma_0\nabla + \partial_t)(e^{-\tau t}v_\tau(x)) = 0$$

and its absolute value has the form

$$\frac{e^{-\tau(t-cx\cdot\omega)}}{\sqrt{\gamma_0(x)}}|1+\epsilon_z(x)| \approx e^{-\tau(t-cx\cdot\omega)}.$$

This again supports the name *virtual slowness* of c.

It is easy to see that the family $(v_\tau)_{\tau\ge C_2}$ satisfies (1.5), (1.6) and (1.7) with $C_2 = ch_D(\omega)$, (1.8) with $C_3 = ch_\Omega(\omega)$ and (1.4) with $C_1 = ch_\Omega(\omega)$. Thus applying Theorem 1.1 to this case, we obtain the following corollary.

Corollary 1.2. *Assume that $\gamma_0 - 1 \in C_0^\infty(\mathbf{R}^n)$. Fix the virtual slowness as $c = C_1$, where C_1 is just the same as Theorem 1.2. Let f be the function of $(x,t) \in \partial\Omega\times]0,\,T[$ having a parameter $\tau > 0$ defined by the equation*

$$f(x,t) = \frac{\partial v}{\partial\nu}(x)\varphi(t),$$

where $v = v_\tau$ is given by (1.29) and a real-valued function $\varphi \in L^2(0,\,T)$ satisfying the condition (1.17) for a $\mu \in \mathbf{R}$. Let $u_f = u_f(x,t)$ be the weak solution of (1.1) for $f = f(x,t;\tau)$. If T satisfies

$$T > 2c(h_\Omega(\omega) - h_D(\omega)), \tag{1.30}$$

then

$$\lim_{\tau\longrightarrow\infty}\frac{1}{2\tau}\log\left|\int_{\partial\Omega}\int_0^T e^{-\tau t}\left(-\overline{v_\tau(x)}f(x,t;\tau) + u_f(x,t)\gamma_0\frac{\partial\overline{v_\tau}}{\partial\nu}(x)\right)dtdS\right| = h_D(\omega).$$

In Corollary 1.2 $c = C_1$ and thus (1.30) should be considered as a *restriction* on the length of the time for data collection. A sufficient condition to ensure (1.30) is

$$T \ge 2C_1(h_\Omega(\omega) + h_\Omega(-\omega)).$$

Note also that the center of coordinates in Theorem 1.2 and Corollary 1.2 is free from D and Ω.

1.3 Real VS Complex

Replace the conditions (1.4), (1.6), (1.7) and (1.8) with the following ones, respectively:

$$\|F(\,\cdot\,,\tau)\|_{L^2(\Omega)}=O(e^{C_1\sqrt{\tau}}\tau^{\mu_1});\tag{1.31}$$

$$\|\nabla v_\tau\|_{L^2(D)}=O(e^{C_2\sqrt{\tau}}\tau^{\mu_2});\tag{1.32}$$

$$\|\nabla v_\tau\|_{L^2(D)}\ge C''e^{C_2\sqrt{\tau}}\tau^{\mu_3};\tag{1.33}$$

$$\|v_\tau\|_{H^1(\Omega)}=O(e^{C_3\sqrt{\tau}}\tau^{\mu_4});\tag{1.34}$$

Then instead of (1.12) we have, for any fixed $T>0$ *without* (1.11) and exactly same g satisfying (1.9) and (1.10) for some $\mu,\mu'\in\mathbf{R}$

$$\lim_{\tau\longrightarrow\infty}\frac{1}{2\sqrt{\tau}}\log\left|\int_{\partial\Omega}\left(g\overline{v}_\tau-w\gamma_0\frac{\partial\overline{v}_\tau}{\partial\nu}\right)dS\right|=C_2.\tag{1.35}$$

Since the proof is simpler than that of Theorem 1.1, we omit its description as a theorem. Explicit examples of v satisfying (1.32), (1.33) and (1.34) in the case when $\gamma_0\equiv 1$ and $n=3$ are the following:

$$v(x;\tau,\omega)=e^{\sqrt{\tau}x\cdot\omega},\ x\in\mathbf{R}^3,\ \omega\in S^2,$$

$$v(x;\tau,p)=\frac{e^{-\sqrt{\tau}|x-p|}}{|x-p|},\ x\in\mathbf{R}^3\setminus\{p\},\ p\in\mathbf{R}^3\setminus\overline{\Omega},$$

$$v(x;\tau,y)=\frac{e^{\sqrt{\tau}|x-y|}-e^{-\sqrt{\tau}|x-y|}}{|x-y|},\ x\in\mathbf{R}^3\setminus\{y\},\ v(y;\tau,y)=2\tau,\ y\in\mathbf{R}^3.$$

These are all real-valued functions and not oscillatory as $\tau\longrightarrow\infty$. We think that this non oscillatory character is an advantage in computing the left-hand side of (1.12).

If g and F are coming from (1.20) and (1.21) for f given by (1.16) for a φ satisfying (1.17) for a $\mu\in\mathbf{R}$, then one can choose

$$C_2=\begin{cases}h_D(\omega), & \text{if } v=v(x;\tau,\omega),\\[2ex] -d_D(p), & \text{if } v=v(x;\tau,p),\\[2ex] R_D(y), & \text{if } v=v(x;\tau,y),\end{cases}$$

where $d_*(p)$ and $R_*(y)$ denote the distance of p to $*$ and minimum radius of the open ball that contains $*$ and centered at y. See Lemma 3.1 in [14] and Proposition 3.2 in [13] for these facts.

By virtue of (1.22) one can choose

$$C_1=\begin{cases}h_\Omega(\omega), & \text{if } v=v(x;\tau,\omega),\\[2ex] -d_\Omega(p), & \text{if } v=v(x;\tau,p),\\[2ex] R_\Omega(y), & \text{if } v=v(x;\tau,y).\end{cases}$$

Then, formula (1.35) reproduces Theorems 1.2-1.4 in [14].

However, when γ_0 is not necessary constant, to construct a suitable v one has to solve the *eikonal equation*

$$\gamma_0 \nabla v \cdot \nabla v = 1 \text{ in } \Omega$$

and corresponding transport equations. However, this is not a simple matter in general because of the complicated behaviour of the characteristic curve $x = x(t)$ under suitable initial conditions on $(x(t), \xi(t))$:

$$\frac{dx}{dt} = 2\xi,$$

$$\frac{d\xi}{dt} = \frac{2}{\sqrt{\gamma_0(x)}} \nabla \left(\frac{1}{\sqrt{\gamma_0(x)}} \right).$$

Thus Corollary 1.2 suggests that when the background body is isotropic, however, not necessary homogeneous, the use of complex geometrical optics solutions are *better* than that of geometrical optics solutions.

2 Proof of Theorem 1.1.

In what follows for simplicity we wright $v_\tau = v$ and $\gamma_0 I_n = \gamma_0$.

Let $R_\gamma(\tau)$ and $R_{\gamma_0}(\tau)$ denote the Neumann-to-Dirichlet maps on $\partial\Omega$ for the operators $\nabla \cdot \gamma \nabla - \tau$ and $\nabla \cdot \gamma_0 \nabla - \tau$, respectively. We have

$$R_{\gamma_0}(\tau) \left(\gamma_0 \frac{\partial \overline{v}}{\partial \nu}|_{\partial\Omega} \right) = \overline{v}|_{\partial\Omega}, \; R_\gamma(\tau) g = p|_{\partial\Omega},$$

where p solves

$$\begin{aligned} &(\nabla \cdot \gamma \nabla - \tau) p = 0 \text{ in } \Omega, \\ &\gamma \nabla p \cdot \nu = g(x, \tau) \text{ on } \partial\Omega. \end{aligned} \tag{2.1}$$

Our starting point is the following identity which is an easy consequence of equations (1.3), (1.5) and integration by parts:

$$\begin{aligned} \int_{\partial\Omega} \left(g\overline{v} - w\gamma_0 \frac{\partial \overline{v}}{\partial \nu} \right) dS &= \int_{\partial\Omega} g(R_{\gamma_0}(\tau) - R_\gamma(\tau)) \left(\gamma_0 \frac{\partial \overline{v}}{\partial \nu}|_{\partial\Omega} \right) dS \\ &+ \int_\Omega (\gamma - \gamma_0) \nabla \overline{v} \cdot \nabla (w - p) dx + e^{-\tau T} \int_\Omega F(x, \tau) \overline{v(x)} dx. \end{aligned} \tag{2.2}$$

Define $\epsilon = w - p$. It follows from (1.3) and (2.1) that ϵ solves

$$(\nabla \cdot \gamma \nabla - \tau)\epsilon = e^{-\tau T} F(x, \tau) \text{ in } \Omega$$

$$\gamma \nabla \epsilon \cdot \nu = 0 \text{ on } \partial\Omega.$$

Since $\tau > 0$, it is easy to see that

$$\|\nabla \epsilon\|_{L^2(\Omega)} \le C e^{-\tau T} \tau^{-1/2} \|F(\,\cdot\,, \tau)\|_{L^2(\Omega)}$$

and from (1.4) one gets

$$\|\nabla\epsilon\|_{L^2(\Omega)} = O(e^{-\tau(T-C_1)}\tau^{\mu_1-1/2}).$$

From this together with (1.4), (1.6) and (1.8) we obtain

$$\begin{aligned}&\int_\Omega(\gamma-\gamma_0)\nabla\overline{v}\cdot\nabla\epsilon dx + e^{-\tau T}\int_\Omega F(x,\tau)\overline{v(x)}dx\\ &= O(e^{C_2\tau}\tau^{\mu_2}e^{-\tau(T-C_1)}\tau^{\mu_1-1/2}) + O(e^{-\tau T}e^{C_3\tau}\tau^{\mu_4}\tau^{\mu_1}e^{C_1\tau})\\ &= O(e^{-\tau(T-C_1-C_2)}\tau^{\mu_1+\mu_2-1/2}) + O(e^{-\tau(T-C_1-C_3)}\tau^{\mu_1+\mu_4}).\end{aligned}\tag{2.3}$$

A combination of (2.2) and (2.3) gives

$$\begin{aligned}\int_{\partial\Omega}\left(g\overline{v}-w\gamma_0\frac{\partial\overline{v}}{\partial\nu}\right)dS &= \Psi(\tau)\int_{\partial\Omega}\gamma_0\frac{\partial v}{\partial\nu}(R_{\gamma_0}(\tau)-R_\gamma(\tau))\left(\gamma_0\frac{\partial\overline{v}}{\partial\nu}|_{\partial\Omega}\right)dS\\ &+O(e^{-\tau(T-C_1-C_2)}\tau^{\mu_1+\mu_2-1/2}) + O(e^{-\tau(T-C_1-C_3)}\tau^{\mu_1+\mu_4}).\end{aligned}\tag{2.4}$$

The following type of estimates now are well known and it is a consequence of Proposition 2.1 in [14] which goes back to [7] and one of assumptions (A1) and (A2). See also [16] when $\gamma_0 = 1$ and $h = (k-1)I_n$ with a positive constant k.

Lemma 2.1. *There exist $C > 0$ and $C' > 0$ such that for all v satisfying (1.5)*

$$C\|\nabla v\|^2_{L^2(D)} \le \left|\int_{\partial\Omega}\gamma_0\frac{\partial v}{\partial\nu}(R_{\gamma_0}(\tau)-R_\gamma(\tau))\left(\gamma_0\frac{\partial\overline{v}}{\partial\nu}|_{\partial\Omega}\right)dS\right| \le C'\|\nabla v\|^2_{L^2(D)}.\tag{2.5}$$

Note also that:

- (1.9) is equivalent to the statement: there exists a $C_0 > 0$ such that, as $\tau\longrightarrow\infty$

$$C_0 \le \tau^\mu|\Psi(\tau)|.\tag{2.6}$$

From the right-hand side of (2.5), (1.6), (2.4), (1.10) and (2.3) we have

$$\begin{aligned}&\left|\int_{\partial\Omega}\left(g\overline{v}-w\gamma_0\frac{\partial\overline{v}}{\partial\nu}\right)dS\right|\\ &= O(\tau^{\mu'}e^{2C_2\tau}\tau^{2\mu_2}) + O(e^{-\tau(T-C_1-C_2)}\tau^{\mu_1+\mu_2-1/2}) + O(e^{-\tau(T-C_1-C_3)}\tau^{\mu_1+\mu_4})\\ &= O(e^{2C_2\tau}\tau^{2\mu_2+\mu'}(1+e^{-\tau(T-C_1+C_2)}\tau^{\mu_1-\mu_2-\mu'-1/2}+e^{-\tau(T-C_1+2C_2-C_3)}\tau^{\mu_1+\mu_4-2\mu_2-\mu'})).\end{aligned}\tag{2.7}$$

One the other hand, from the left-hand side of (2.5), (2.6) and (1.7) we obtain

$$\begin{aligned}&\left|\int_{\partial\Omega}\left(g\overline{v}-w\gamma_0\frac{\partial\overline{v}}{\partial\nu}\right)dS\right|\\ &\ge CC_0(C'')^2\tau^{-\mu}e^{2C_2\tau}\tau^{2\mu_3} + O(e^{-\tau(T-C_1-C_2)}\tau^{\mu_1+\mu_2-1/2}) + O(e^{-\tau(T-C_1-C_3)}\tau^{\mu_1+\mu_4})\\ &= C'''e^{2C_2\tau}\tau^{2\mu_3-\mu}(1+O(e^{-\tau(T-C_1+C_2)}\tau^{\mu_1+\mu_2-2\mu_3+\mu-1})+O(e^{-\tau(T-C_1+2C_2-C_3)}\tau^{\mu_1+\mu_4-2\mu_3+\mu})),\end{aligned}\tag{2.8}$$

where $C''' = CC_0(C'')^2 > 0$.

Now formula (1.12) is a consequence of (1.11), (2.7) and (2.8) provided

$$T > \max(C_1 - C_2, C_1 + C_3 - 2C_2). \tag{2.9}$$

However, by Remark 1.2 we have $(C_1 + C_3 - 2C_2) - (C_1 - C_2) = C_3 - C_2 \geq 0$ and thus (2.9) is nothing but (1.11).

□

3 Proof of Theorem 1.2

3.1 A special fundamental solution

Given F we construct a solution of the inhomogeneous modified Helmholtz equation

$$(-\triangle + \tau)v + F = 0 \text{ in } \mathbf{R}^n \tag{3.1}$$

that has the form

$$v(x) = e^{x\cdot z}\Psi(x) \tag{3.2}$$

where the complex vector z is given by (1.13).

Write

$$F(x) = e^{x\cdot z}f(x). \tag{3.3}$$

Then if Ψ satisfies the equation

$$-\triangle\Psi - 2z\cdot\nabla\Psi + f = 0 \text{ in } \mathbf{R}^n, \tag{3.4}$$

then the v given by (3.2) satisfies (3.1) with F given by (3.3).

We construct a solution of (3.4) by using a special fundamental solution of (3.4). Set

$$Q_z(\xi) = |\xi|^2 - 2iz\cdot\xi,\ \xi\in\mathbf{R}^n.$$

We have

$$\mathrm{Re}\, Q_z(\xi) = \left|\xi + c\tau\sqrt{1-\frac{1}{c^2\tau}}\,\omega^{\perp}\right|^2 - c^2\tau^2\left(1-\frac{1}{c^2\tau}\right)$$

and

$$\mathrm{Im}\, Q_z(\xi) = -2c\tau\omega\cdot\xi.$$

Thus the set $Q_z^{-1}(0) = \{\xi\in\mathbf{R}^n\,|\,Q_z(\xi)=0\}$ consists of the circle on the plane $\omega\cdot\xi = 0$ centered at $-c\tau\sqrt{1-(c^2\tau)^{-1}}\omega^{\perp}$ with radius $c\tau\sqrt{1-(c^2\tau)^{-1}}$. Moreover $\nabla\mathrm{Re}\, Q_z(\xi)$ and $\nabla\mathrm{Im}\, Q_z(\xi)$ on $Q_z^{-1}(0)$ are linearly independent. Thus $-1/Q_z(\xi)$ is locally integrable on $\mathbf{R}^n$ and can be identified with a unique tempered distribution and its inverse Fourier transform

$$g_z(x) = -\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\frac{e^{ix\cdot\xi}}{Q_z(\xi)}d\xi \tag{3.5}$$

is well defined. This $g = g_z$ satisfies, in the sense of tempered distribution

$$-\triangle g - 2z\cdot\nabla g + \delta(x) = 0 \text{ in } \mathbf{R}^n$$

and thus, in the sense of Schwartz distribution

$$(-\triangle+\tau)(e^{x\cdot z}g_z)+\delta(x)=0 \text{ in } \mathbf{R}^n.$$

The $G_z(x)=e^{x\cdot z}g_z(x)$ should be called the Faddeev-Green function.

From [19], one knows that: by virtue of Theorem 7.1.27 of [5] given $f\in L_{\delta+1}(\mathbf{R}^n)$ with $-<1<\delta<0$ the solution of equation (3.4) is unique in $L^2_\delta(\mathbf{R}^n)$; if f is a rapidly decreasing function, then the unique solution of (3.4) is given by

$$\Psi=g_z*f; \tag{3.6}$$

its first and second derivatives belong to $L^2_\delta(\mathbf{R}^n)$ and satisfy

$$\|D^\alpha\Psi\|_{L^2_\delta(\mathbf{R}^n)}\le C_\alpha(z,\delta)\|f\|_{L^2_{1+\delta}(\mathbf{R}^n)},\ |\alpha|\le 2. \tag{3.7}$$

Thus the operator $:f\longmapsto\Psi$ has the unique continuous extension as a bounded linear operator of $L^2_{\delta+1}(\mathbf{R}^n)$ to $L^2_\delta(\mathbf{R}^n)$. We still denote the operator by the same symbol g_z*f which yields the unique solution of equation (3.4) in $L^2_\delta(\mathbf{R}^n)$ for *all* $f\in L^2_{\delta+1}(\mathbf{R}^n)$.

However, for our purpose, one has to clarify the behaviour of $C_\alpha(z,\delta)$ in (3.7) as $\tau\longrightarrow\infty$. This is not the well known case $z\cdot z=-k^2$ with a $k\ge 0$ which appeared in inverse scattering/boundary value problems (cf.[19, 17, 18]) since we have (1.14) and thus $z\cdot z\longrightarrow\infty$ as $\tau\longrightarrow\infty$. In the next subsection we study more about $C_\alpha(z,\delta)$ by carefully checking a proof of (3.7).

3.2 Asymptotic behaviour

Define

$$\lambda(c,\tau)=\sqrt{1-\frac{1}{c^2\tau}}.$$

The aim of this subsection is to give a proof of the following estimates.

Proposition 3.1. *Let $R>0$ and $-1<\delta<0$. We have*

$$\|D^\alpha g_z*f\|_{L^2_\delta(\mathbf{R}^n)}\le (c\tau\lambda)^{|\alpha|-1}C_{\delta,R}\|f\|_{L^2_{\delta+1}(\mathbf{R}^n)},\ |\alpha|\le 2,\ c\tau\lambda\ge R. \tag{3.8}$$

A change of variables yields

$$g_z(x)=(c\tau\lambda(c,\tau))^{n-2}h_\lambda(y;\omega,\omega^\perp)|_{y=c\tau\lambda(c,\tau)x,\ \lambda=\lambda(c,\tau)} \tag{3.9}$$

where

$$h_\lambda(y;\omega,\omega^\perp)=-\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\frac{e^{iy\cdot\xi}d\xi}{|\xi+\omega^\perp|^2-1-2i\lambda^{-1}\omega\cdot\xi},\ 0<\lambda\le 1. \tag{3.10}$$

Given an arbitrary rapidly decreasing function f we give an estimate of $h_\lambda*f$. Using a rotation in ξ-space, it suffices to consider the case when $\omega=(0,1,0,\cdots,0)^T$ and $\omega^\perp=(1,0,\cdots,0)^T$.

Define

$$\Sigma=\{\xi\in\mathbf{R}^n\,|\,|\xi+\omega^\perp|^2=1,\ \omega\cdot\xi=0\}.$$

Σ is the set of all real zero points of the complex-valued polynomial

$$h(\xi;\lambda) = |\xi + \omega^{\perp}|^2 - 1 - 2i\lambda^{-1}\omega\cdot\xi,$$

however, Σ itself is *independent* of λ.

Lemma 3.1. *Given $\epsilon > 0$ there exists a positive constant C_ϵ independent of λ such that for all $\xi \in \mathbf{R}^n$ with $dist_\Sigma(\xi) \geq \epsilon$, we have*

$$|h(\xi;\lambda)| \geq C_\epsilon|\xi|^2.$$

Proof. Since $0 < \lambda \leq 1$, we have

$$|h(\xi;\lambda)| \geq |h(\xi;1)|.$$

Let $|\xi| \geq 2$. We have

$$|h(\xi;1)|^2 = (|\xi|^2 + 2\omega^{\perp}\cdot\xi)^2 + 4(\omega\cdot\xi)^2$$

$$= |\xi|^4 + 4(\omega^{\perp}\cdot\xi)|\xi|^2 + 4(\omega^{\perp}\cdot\xi)^2 + 4(\omega\cdot\xi)^2$$

$$\geq |\xi|^4 - 4|\xi|^3 = |\xi|^4\left(1 - \frac{1}{|\xi|}\right) \geq \frac{|\xi|^4}{2}.$$

Since $\xi \neq 0$ and $|h(\xi;1)| > 0$ for ξ with $\mathrm{dist}_\Sigma(\xi) \geq \epsilon$, the function $|\xi|^{-2}|h(\xi;1)|$ is continuous and positive on the nonempty compact set $K_\epsilon = \{\xi \in \mathbf{R}^n \mid |\xi| \leq 2 \,\text{and}\, \mathrm{dist}_\Sigma(\xi) \geq \epsilon\}$. Thus $m_\epsilon \equiv \inf_{\xi\in K_\epsilon} |\xi|^{-2}|h(\xi;1)| > 0$ and choosing

$$C_\epsilon = \min\left(\frac{1}{\sqrt{2}}, m_\epsilon\right),$$

we obtain the desired estimate.
□

For the treatment of $h(\xi;\lambda)$ in a neighbourhood of Σ, we start with the following fundamental fact:

$$-\frac{1}{2\pi i(x_1 + ix_2)} = \frac{1}{(2\pi)^2}\int \frac{e^{ix\cdot\eta}}{\eta_1 + i\eta_2}d\eta.$$

This yields

$$-\frac{1}{2\pi i(\lambda^{-1}x_1 + ix_2)} = \frac{1}{(2\pi)^2}\int \frac{e^{ix\cdot\eta}}{\eta_1 + i\lambda^{-1}\eta_2}d\eta. \tag{3.11}$$

Lemma 3.2. *Let $-1 < \delta < 0$ and $0 < \lambda \leq 1$. Then*

$$Z_\lambda f = \left(\frac{\hat{f}}{\eta_1 + i\lambda^{-1}\eta_2}\right)^{\vee}$$

defines a bounded map from $L^2_{\delta+1}(\mathbf{R}^n)$ to $L^2_\delta(\mathbf{R}^n)$ and its operator norm is bounded from above with respect to λ.

Proof. From (3.11) we have

$$Z_\lambda f = -\frac{1}{2\pi i}\left\{\frac{1}{\lambda^{-1}x_1 + ix_2}\right\} * f,$$

where $*$ denotes convolution with respect to variables (x_1, x_2). Let g be an arbitrary rapidly decreasing function. Since $0 < \lambda \le 1$ we have

$$|\lambda^{-1}x_1 + ix_2| \ge |x|.$$

This gives

$$(2\pi)^2| < Z_\lambda f, g > |^2 \le \left(\int\int\left|\frac{g(x)f(y)}{\lambda^{-1}(x_1 - y_1) + i(x_2 - y_2)}\right| dxdy\right)^2$$

$$\le \left(\int\int\left|\frac{g(x)f(y)}{(x_1 - y_1) + i(x_2 - y_2)}\right| dxdy\right)^2.$$

Thus everything is reduced to the case when $\lambda = 1$ and it is nothing but Lemma 3.1 in [19].
□

Having Lemmas 3.1-2 and using a rotation in ξ-space, localization and a change of variable which are exactly same as Sylvester-Uhlmann's argument [19] for the case $z\cdot z = 0$, we obtain

$$\|D^\alpha h_\lambda * f\|_{L^2_\delta(\mathbf{R}^n)} \le C_\delta\|f\|_{L^2_{\delta+1}(\mathbf{R}^n)},\ |\alpha| \le 2. \tag{3.12}$$

The constant C_δ is independent of λ.

To deduce (3.8) from (3.12) we employ a scaling argument [6]. It follows from (3.9) that

$$g_z * f = (c\tau\lambda)^{-2}\left(h_\lambda * f_{(c\tau\lambda)^{-1}}\right)_{c\tau\lambda},\ \lambda = \lambda(c, \tau), \tag{3.13}$$

where

$$f_\eta(x) = f(\eta x),\ \eta > 0.$$

Let $s > 0$. We have

$$\|f_{\eta^{-1}}\|_{L^2_s(\mathbf{R}^n)} \le \eta^{s+n/2}\max(1, R^{-s})\|f\|_{L^2_s(\mathbf{R}^n)},\ 0 < R \le \eta.$$

Now from this and (3.13) we have

$$| < g_z * f, g > | = (c\tau\lambda)^{-2}| < (h_\lambda * f_{(c\tau\lambda)^{-1}})_{c\tau\lambda}, g > |$$

$$= (c\tau\lambda)^{-(2+n)}| < h_\lambda * f_{(c\tau\lambda)^{-1}}, g_{(c\tau\lambda)^{-1}} > |$$

$$\le (c\tau\lambda)^{-(2+n)}\|h_\lambda * f_{(c\tau\lambda)^{-1}}\|_{L^2_\delta(\mathbf{R}^n)}\|g_{(c\tau\lambda)^{-1}}\|_{L^2_{-\delta}(\mathbf{R}^n)}$$

$$\le (c\tau\lambda)^{-(2+n)}C_\delta\|f_{(c\tau\lambda)^{-1}}\|_{L^2_{\delta+1}(\mathbf{R}^n)}\|g_{(c\tau\lambda)^{-1}}\|_{L^2_{-\delta}(\mathbf{R}^n)}$$

$$\le (c\tau\lambda)^{-(2+n)}C_\delta(c\tau\lambda)^{\delta+1+n/2}\max(1, R^{-(\delta+1)})\|f\|_{L^2_{\delta+1}(\mathbf{R}^n)}(c\tau\lambda)^{-\delta+n/2}\max(1, R^\delta)\|g\|_{L^2_{-\delta}(\mathbf{R}^n)}$$

$$= (c\tau\lambda)^{-1}C(\delta, R)\|f\|_{L^2_{\delta+1}(\mathbf{R}^n)}\|g\|_{L^2_{-\delta}(\mathbf{R}^n)}.$$

This together with the same argument for $D^\alpha g_z * f$ yields (3.8).

3.3 Uniqueness and Construction of ϵ_z

Write (1.24) as

$$(-\triangle - 2z\cdot\nabla)\epsilon_z = (\tau a + b)\epsilon_z - (\tau a + b) \text{ in } \mathbf{R}^n.$$

Since the solution of (3.4) is unique in $L^2_\delta(\mathbf{R}^n)$ and has the form (3.6) for $f \in L^2_{\delta+1}(\mathbf{R}^n)$, we have

$$\epsilon_z = \tau g_z * (a\epsilon_z) - \tau g_z * a + g_z * (b\epsilon_z) - g_z * b. \tag{3.14}$$

Define

$$K_z(a) : L^2_\delta(\mathbf{R}^n) \ni h \longmapsto \tau g_z * (ah) \in L^2_\delta(\mathbf{R}^n)$$

and

$$L_z(b) : L^2_\delta(\mathbf{R}^n) \ni h \longmapsto g_z * (bh) \in L^2_\delta(\mathbf{R}^n).$$

It follows from (3.7) that both K_z and L_z define bounded linear operators in $L^2_\delta(\mathbf{R}^n)$. Rewrite (3.14) as

$$(I - K_z(a) - L_z(b))\epsilon_z = -\tau g_z * a - g_z * b.$$

It follows from (3.8) for $|\alpha| = 0$ that

$$\|K_z(a)h\|_{L^2_\delta(\mathbf{R}^n)} \le (c\lambda)^{-1} C_{\delta,R} \|ah\|_{L^2_{\delta+1}(\mathbf{R}^n)}$$

provided $c\lambda \ge R > 0$. Since $\|ah\|_{L^2_{\delta+1}(\mathbf{R}^n)} \le \| <x> a\|_{L^\infty(\mathbf{R}^n)} \|h\|_{L^2_\delta(\mathbf{R}^n)}$, one gets

$$\|K_z(a)\| \le (c\lambda)^{-1} C_{\delta,R} \| <x> a\|_{L^\infty(\mathbf{R}^n)}$$

and similarly

$$\|L_z(b)\| \le (c\tau\lambda)^{-1} C_{\delta,R} \| <x> b\|_{L^\infty(\mathbf{R}^n)}.$$

From these we obtain

$$\|K_z(a) + L_z(b)\| \le (c\lambda)^{-1} C(\delta, R)(\| <x> a\|_{L^\infty(\mathbf{R}^n)} + \tau^{-1} \| <x> b\|_{L^\infty(\mathbf{R}^n)}). \tag{3.15}$$

Let $c_1 \ge c_2 > 0$ and $R > 0$. Let c and τ satisfy

$$c \ge \sqrt{c_1^2 + R^2}$$

and

$$\tau \ge \frac{1}{c_2^2}.$$

We have $c^2\tau > 1$ and $c\lambda \ge R$. Now given $0 < \delta < 1$, $R > 0$ and $\eta > 0$ choose a large c_1 in such a way that

$$\eta\sqrt{c_1^2 - c_2^2} \ge C(\delta, R)(\| <x> a\|_{L^\infty(\mathbf{R}^n)} + c_2^2 \| <x> b\|_{L^\infty(\mathbf{R}^n)}). \tag{3.16}$$

Since $c\lambda \ge \sqrt{c_1^2 - c_2^2}$, a combination of (3.15) and (3.16) gives

$$\|K_z(a) + L_z(b)\| \le \eta. \tag{3.17}$$

Choosing a $\eta < 1$ and

$$C_1 = \sqrt{c_1^2 + R^2},\ C_2 = 1/c_2^2$$

for c_1 satisfying (3.16), specially chosen $c_2 \leq c_1$ and R, we obtain the uniqueness and existence of the solution of (3.12) and thus those of (1.27), too.

$\epsilon_z \in L^2_\delta(\mathbf{R}^n)$ takes the form

$$\epsilon_z = -\sum_{n=0}^{\infty}(K_z(a) + L_z(b))^n(\tau g_z * a + g_z * b).$$

It follows from (3.17) and (3.8) for $|\alpha| = 0$ that

$$\begin{aligned}\|\epsilon_z\|_{L^2_\delta(\mathbf{R}^n)} &\leq \sum_{n=0}^{\infty}\eta^n(c\lambda)^{-1}C(\delta,R)(\|a\|_{L^2_{\delta+1}(\mathbf{R}^n)} + c_2^2\|b\|_{L^2_{\delta+1}(\mathbf{R}^n)}) \\ &= \frac{1}{1-\eta}(c\lambda)^{-1}C(\delta,R)(\|a\|_{L^2_{\delta+1}(\mathbf{R}^n)} + c_2^2\|b\|_{L^2_{\delta+1}(\mathbf{R}^n)}) = O((c\lambda)^{-1}).\end{aligned} \tag{3.18}$$

Differentiating both sides of (3.14) in the sense of distribution, we have

$$(I - K_z(a) - K_z(b))\partial_j\epsilon_z = -\tau g_z * \partial_j a - K_z(\partial_j a)\epsilon_z - g_z * \partial_j b - L_z(\partial_j b)\epsilon_z.$$

A similar argument for the derivation of (3.18) yields

$$\|\partial_j\epsilon_z\|_{L^2_\delta(\mathbf{R}^n)} = O((c\lambda)^{-1}).$$

Continue this procedure for higher order derivatives of ϵ_z in $L^2_\delta(\mathbf{R}^n)$ and apply the Sobolev imbedding theorem to the resulted estimates. Then one concludes that $\epsilon_z|_\Omega$ can be identified with a function in $C^1(\overline{\Omega})$ and

$$\|\epsilon_z\|_{L^\infty(\Omega)} + \|\nabla\epsilon_z\|_{L^\infty(\Omega)} = O((c\lambda)^{-1}).$$

Thus choosing again a large c_1, we obtain (1.28). This completes the proof of Theorem 1.2.

Acknowledgement

This research was partially supported by Grant-in-Aid for Scientific Research (C)(No. 21540162) of Japan Society for the Promotion of Science.

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